Real Geometry of Matrix Completion

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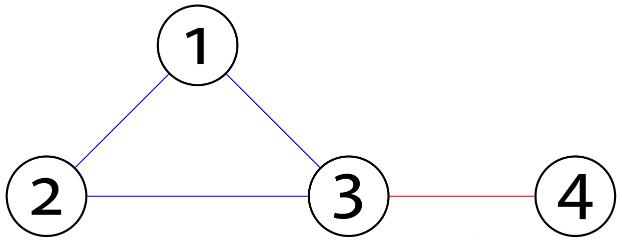
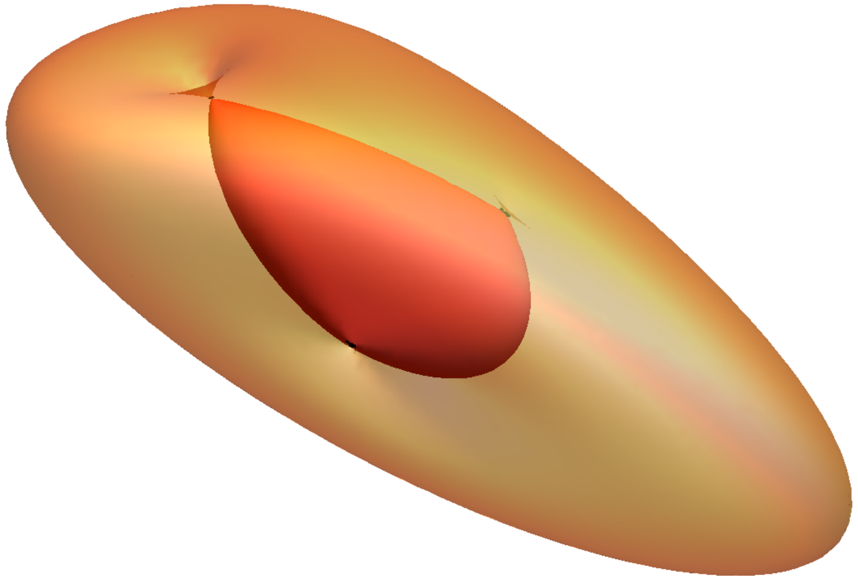
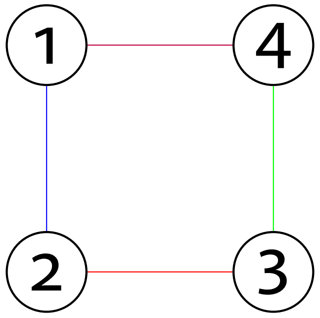






Symmetric matrix completion

$$\begin{pmatrix} 1 & ? & -3 \\ ? & 7 & ? \\ -3 & ? & -2 \end{pmatrix}$$

Constraints:

(1) Rank (smallest possible, fixed, …)

(2) Positive semidefinite

$$\begin{pmatrix} * & ? & * \\ ? & * & ? \\ * & ? & * \end{pmatrix}$$

Spectral Theorem

Definition. A real symmetric matrix is positive semidefinite if all of its **real** eigenvalues are nonnegative.

Theorem. *A real symmetric matrix M is positive semidefinite if and only if there exists a real matrix B such that $M = B^T B$.*

Real symmetric matrices M are in one-to-one correspondence with real quadratic forms $Q_M = (x_1, x_2, \ldots, x_n) M (x_1, x_2, \ldots, x_n)^T$.

Example.

$$M = \begin{pmatrix} 1 & 7 & -2 & 3 \\ 7 & 1 & 5 & -13 \\ -2 & 5 & 1 & 11 \\ 3 & -13 & 11 & 1 \end{pmatrix}$$

$$Q_M = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 14x_1x_2 - 4x_1x_3 + 6x_1x_4 + 10x_2x_3 - 26x_2x_4 + 22x_3x_4$$

Nonnegative Quadratic Forms and Sums of Squares

Theorem. *A quadratic form $Q_M = x^T M x$ is positive semidefinite if and only if it is a sum of squares $Q_M = (Bx)^T(Bx)$.*

Example.

$$74x_1^2+122x_2^2+173x_3^2+144x_1x_2+202x_1x_3+282x_2x_3 = (5x_1-x_2+2x_3)^2+(7x_1+11x_2-13x_3)^2$$

$$\begin{pmatrix} 74 & 72 & 101 \\ 72 & 122 & 141 \\ 101 & 141 & 173 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ -1 & 11 \\ 2 & -13 \end{pmatrix} \begin{pmatrix} 5 & -1 & 2 \\ 7 & 11 & -13 \end{pmatrix}$$

Smallest number of squares in $Q_M = \ell_1^2 + \ldots + \ell_r^2$ = rank of M

Obvious Necessary Condition

Theorem (Sylvester's criterion). *A symmetric matrix is positive semidefinite if and only if all of its principal minors are nonnegative.*

Example.

$$M = \begin{pmatrix} 1 & 2 & ? \\ 2 & 1 & ? \\ ? & ? & 7 \end{pmatrix}$$

$$Q_M = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + ?x_1x_3 + ?x_2x_3$$

$$Q_M(1, -1, 0) = 1 + 1 - 4 = -2$$

Example.

$$\begin{pmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{pmatrix}$$

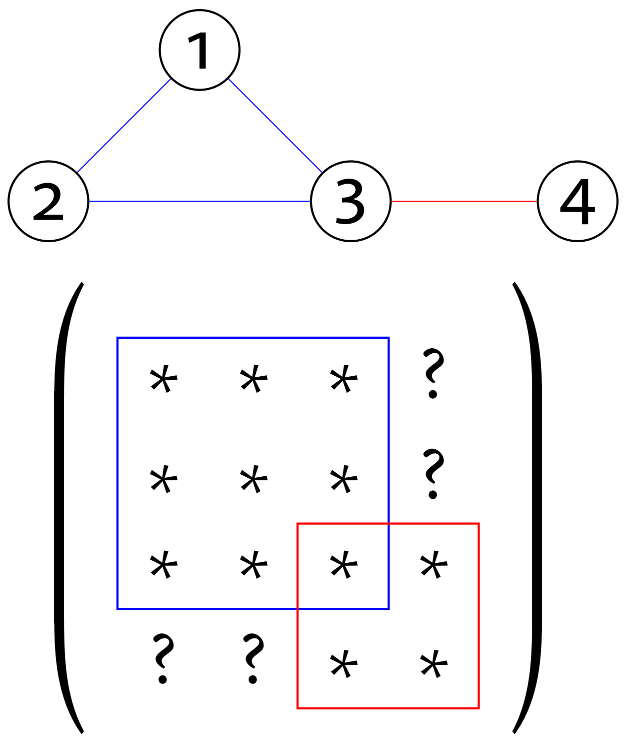
Aside: diagonal entries

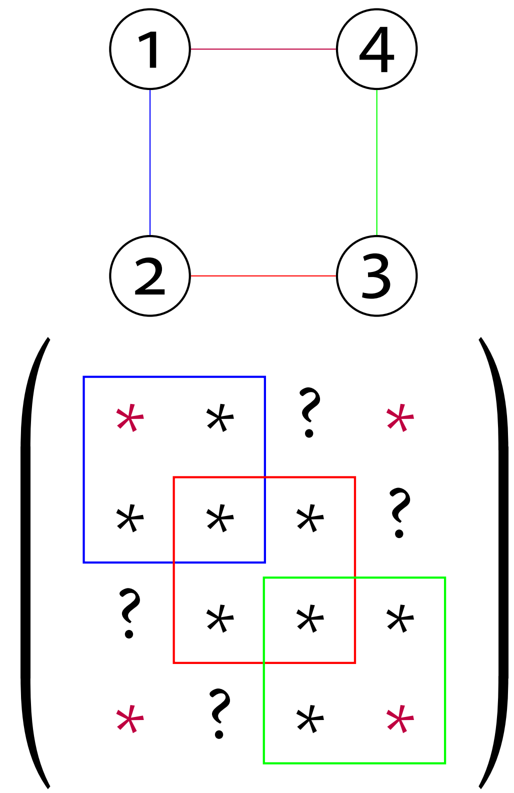
$$\begin{pmatrix} 1 & ? & ? \\ ? & 2 & 1 \\ ? & 1 & ? \end{pmatrix}$$

$$\begin{pmatrix} 1 & ? & ? \\ ? & 0 & 1 \\ ? & 1 & ? \end{pmatrix}$$

Patterns and graphs

Complete principal minors: Cliques





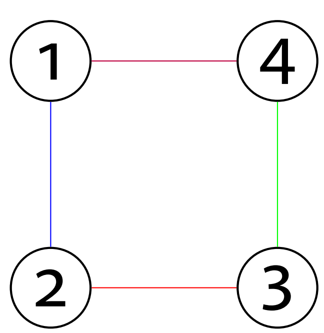
Stanley-Reisner Ideals

Definition. The **Stanley-Reisner ideal** associated to $G = ([n], E)$ is the square-free monomial ideal

$$I_G = \langle x_i x_j \colon \{i, j\} \notin E \rangle \subset \mathbb{R}[x_1, x_2, \ldots, x_n].$$

Proposition. *(1) G-partial matrices M are in one-to-one correspondence with residue classes of quadratic forms Q_M in $\mathbb{R}[x_1, x_2, \ldots, x_n]/I_G = \mathbb{R}[X_G]$.*
(2) A G-partial matrix M has a positive semidefinite completion if and only if the corresponding quadratic form Q_M is a sum of squares of linear forms modulo I_G.

Example.



$$I_G = \langle x_1 x_3, x_2 x_4 \rangle \qquad M = \begin{pmatrix} 1 & 7 & ? & 3 \\ 7 & 1 & 5 & ? \\ ? & 5 & 1 & 11 \\ 3 & ? & 11 & 1 \end{pmatrix}$$

$$Q_M = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 14x_1x_2 + 6x_1x_4 + 10x_2x_3 + 22x_3x_4 + ?x_1x_3 + ?x_2x_4$$

Minimal Free Resolutions

Let $S = \mathbb{R}[x_1, x_2, \ldots, x_n]$.

Definition. A **minimal free resolution** of a graded S-module M is an exact sequence

$$\cdots \xrightarrow{\delta_{t+1}} F_t \xrightarrow{\delta_t} F_{t-1} \xrightarrow{\delta_{t-1}} \cdots \xrightarrow{\delta_1} F_0 \to M \to 0,$$

where the F_i are free S-modules and the image of the map δ_i is contained in the submodule $(x_1, x_2, \ldots, x_n)F_{i-1}$ of F_{i-1} for all $i \geq 1$.

Example. *Consider $M_1 = \langle x_1x_4, x_2x_4 \rangle$ and $M_2 = \langle x_1x_3, x_2x_4 \rangle$.*

$$0 \to S \xrightarrow{(x_2, -x_1)^T} S^2 \xrightarrow{(x_1x_4, x_2x_4)} M_1 \to 0$$

$$0 \to S \xrightarrow{(x_2x_4, -x_1x_3)^T} S^2 \xrightarrow{(x_1x_3, x_2x_4)} M_2 \to 0$$

are minimal free resolutions of M_1 and M_2.

Definition. A homogeneous ideal $I \subset S$ is 2-**regular** (Castelnuovo-Mumford) if it is generated by quadrics and the entries of δ_i have degree 1 for all $i \geq 1$.

Theorem (Fröberg). *The ideal I_G is 2-regular if and only if G is chordal.*

The Hankel Spectrahedron

Definition. The convex dual cone of Σ_G is the **Hankel spectrahedron** of G.

Proposition.

$$\Sigma_G^\vee = \mathcal{S}_+ \cap (\ker(\pi_G))^\perp$$

Proof.

$$\Sigma_G^\vee = (\pi_G(\mathcal{S}_+))^\vee = \mathcal{S}_+ \cap (\ker(\pi_G))^\perp$$

Question: What are the extreme rays of $\Sigma_G^\vee$?

Proposition. *Extreme rays of $\Sigma_G^\vee$ that have rank 1 are in one-to-one correspondence with points $x \in \mathcal{V}(I_G) = X_G$.*

Proof. $R = xx^T \in \Sigma_G^\vee$, $M \in \ker(\pi_G)) \cong I_G$:

$$0 = \langle M, R\rangle = \mathrm{tr}(MR) = \mathrm{tr}(Mxx^T) = x^T Mx = Q_M(x)$$

Extreme Rays of the Dual Convex Cone

Proposition. *Let $G = ([m], E)$ be the m-cycle. The matrix*

$$\begin{pmatrix} \frac{m-2}{m-1} & -1 & 0 & 0 & \dots & 0 & 0 & \frac{1}{m-1} \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & \vdots & \vdots & \vdots \\ \vdots & 0 & -1 & 2 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ \frac{1}{m-1} & 0 & 0 & 0 & \dots & 0 & -1 & \frac{m-2}{m-1} \end{pmatrix}$$

$m = 4$

is an extreme ray of $\Sigma_G^\vee$. It has rank $m-2$. It certifies that

$$\begin{pmatrix} 1 & 1 & ? & ? & \dots & ? & -1 \\ 1 & 1 & 1 & ? & \dots & ? & ? \\ ? & 1 & 1 & 1 & \dots & ? & ? \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ ? & ? & ? & ? & \dots & 1 & 1 \\ -1 & ? & ? & ? & \dots & 1 & 1 \end{pmatrix}$$

does not have a positive semidefinite completion even though every complete principal minor is positive semidefinite.

Hankel index and bpf linear series

Definition. The **Hankel index** of G is the smallest $r > 1$ such that $\Sigma_G^\vee$ has an extreme ray of rank r.

Let $R \in \Sigma_G^\vee = \mathcal{S}_+ \cap (\ker(\pi_G))^\perp$ be an extreme ray.

(1) The kernel of R is a linear series on X_G: $\ker(R) \subset \mathbb{R}[x_1, x_2, \ldots, x_n]_1$.

(2) If $\mathrm{rk}(R) > 1$, then $\ker(R)$ is a base-point free linear series on X_G. (Blekherman)

(3) $\langle \ker(R) \rangle_2$ is contained in $R^\perp$.

Theorem (Blekherman, Sinn, Velasco). *The Hankel index of G is at least the (Green-Lazarsfeld index of I_G) + 1.*

Theorem (Eisenbud, Green, Hulek, Popescu). *The Green-Lazarsfeld index of I_G is the (smallest length of a chordless cycle in G) - 2.*

Theorem. *Let m the smallest length of a chordless cycle in G. A G-partial matrix M has a positive definite completion if and only if*

(1) every complete principal minor of M is positive, and

(2) M has a positive semidefinite completion of rank at least $n - m + 2$.

MLT vs. GCR

Let $G = ([n], E)$ be a labeled simple graph.

Definition. (1) The **maximum likelihood threshold** of G is the smallest whole number k such that there exists a positive definite matrix D with $\pi_G(M) = \pi_G(D)$ for almost all positive semidefinite matrices M of rank at most k.
(2) The **generic completion rank** of G is the smallest whole number k such that $\dim(\pi_G(V_k)) = n + \#E$, where V_k is the variety of matrices of rank at most k.

Proposition (Sard's Theorem and Implicit Function Theorem). $\mathrm{gcr}(G) \geq \mathrm{mlt}(G)$

Theorem (Blekherman, Sinn).

$$\mathrm{gcr}(K_{m,m}) = m$$
$$\mathrm{mlt}(K_{m,m}) = \min\left\{k \colon \binom{k+1}{2} \geq 2m\right\} \in O(2\sqrt{m}).$$

Gram Dimension (SOS length)

Definition (Laurent, Varvitsiotis). The **Gram dimension** of a labeled simple graph $G = ([n], E)$ is the smallest whole number k such that for any positive semidefinite matrix $M \in \mathcal{S}_+$, there exists another $M' \in \mathcal{S}_+$ of rank at most k such that $\pi_G(M) = \pi_G(M')$.

Theorem (Laurent, Varvitsiotis). *(1) $k \leq 3$: The Gram dimension of G is at most k if and only if G has no K_{k+1} minor.*
(2) The Gram dimension of G is at most 4 if and only if G has no K_5 and $K_{2,2,2}$ minors.

Example. *The Gram dimension of any m-cylce is 3.*